UNIT-4

Statistical Techniques II

overview of Probability Random variables (Discrete and continuous Random variable) Probability mass function and Probability density function, Expectation and Variance, Discrete and continuous probability distribution: Binomial, Poission and Normal distributions.

Random Experiment: occurences which can be repeated a number of times, essentially under the same conditions, and whose result can not be predicted before hand are known as random experiments.

e.g → rolling of a die, tossing a coin, taking out balls from an urn.

Sample space: out of the several possible outcomes of a random experiment, one and only one can take place in a trial. The set of all these possible outcomes is called the sample space for the particular experiment and is denoted by S.

e.g → if a coin is tossed, the possible outcomes are H (Head) and T (Tail) Thus $S = \{H, T\}$

Sample Point: The elements of S, the sample space, are called sample points.

Event:- Every subset of S, the sample space, is called an event.

since $S \subset S$, S is called certain event

$\phi \subset S$, the null set is called Impossible event.

Random Variable

A Real valued function X, defined on a sample space S of a random experiment E is called a random variable which assigns to each sample point $s \in S$ one & only one real number $X(s) = x$; where x is any real number.

Random Variable $X: S \rightarrow \mathbb{R}$, $X(s) = x \in \mathbb{R}, \forall s \in$

The Domain of the random variable X is the Sample Space S and Range is non-empty set of real no.

Example:- If three coins are tossed, then the sample space contains 8 sample points. Let the random variable X denote "the number of heads". Then X is a real valued function over S with a space A which has four elements 0, 1, 2, 3 as

Sample Point	$X(s)$	X = No. of heads	Probability
TTT	$X(TTT)$	0	1/8
HTT, THT, TTH	$X(HTT), X(THT), X(TTH)$	1	3/8
HHT, HTH, THH	$X(HHT), X(HTH), X(THH)$	2	3/8
HHH	$X(HHH)$	3	1/8

Thus A be the space of X is a set of real numbers given by $A = \{x : x = X(s), s \in S\}$

$S = \{TTT, HTT, THT, TTH, THH, HTH, HHT, HHH\}$

$A = \{0, 1, 2, 3\}$

Types of Random variable :-

1. Discrete Random variable
2. continuous Random variable

1. Discrete Random variable -

A discrete Random variable is one which can assume only a discrete set of values or isolated values.

e.g → the number of heads in 4 tosses of a coin is a discrete random variable as it can not assume values other than 0, 1, 2, 3, 4

Probability Mass function (PMF)

let $x_1, x_2, \dots, x_n$ be the values of discrete random variable X and let $P_1, P_2, \dots, P_n$ (where $P_i \geq 0$) be corresponding probability Then a function $f(x)$ or $p(x)$ defined by

$$P(X=x) = p(x) = \begin{cases} p(x_i) \text{ or } p_i, & x = x_i \quad i = 1, 2, \dots \\ 0 & \text{Otherwise} \end{cases}$$

is called the Probability of Mass function of the discrete random variable X.

Note :- i.) $p(x_i) \geq 0$ ii.) $\sum p(x_i) = 1$

Example :- A Random variable X takes values $1, 2, 3, \dots$ with Probability mass function $\frac{\lambda^r}{r!}$, $r = 1, 2, 3, \dots \infty$

Find λ.

Soln :- $\sum_{r=1}^{\infty} \frac{\lambda^r}{r!}$ should be 1. Thus

$$\sum_{r=1}^{\infty} \frac{\lambda^r}{r!} = 1 \Rightarrow \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots = 1$$

$$\Rightarrow e^{\lambda} - 1 = 1 \Rightarrow e^{\lambda} = 2 \Rightarrow \boxed{\lambda = \log 2}$$

Probability distribution of a discrete Random variable :-

A table or a formula listing all possible values that random variable can take on together with the respective Probabilities, is called a Probability distribution of the random variable.

or The set of ordered pairs $[x_i, p(x_i)]$ is called the Probability distribution of a discrete random variable X Provided $p(x_i) \geq 0$ and $\sum p(x_i) = 1$.

e.g → Suppose a coin is tossed three times. Then the Probability distribution of the no. of heads is

$X = x$:	0	1	2	3
$P(X=x) = p(x)$:	$1/8$	$3/8$	$3/8$	$1/8$

Mean and Variance of Random variables :-

Let X : $x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n$
$P(X)$: $p_1 \quad p_2 \quad p_3 \quad \dots \quad p_n$
be discrete Probability distribution.

Then

$$\text{Mean } \mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i \qquad [\because \sum p_i = 1]$$

$$\text{Variance } \sigma^2 = \sum p_i (x_i - \mu)^2$$

$$\sigma^2 = \sum p_i x_i^2 - \mu^2$$ if μ is not whole no.

Standard deviation $\sigma = +\sqrt{\text{Variance}}$

ample: 1 Five Defective bulbs are accidentally mixed with twenty good ones. It is not Possible to just look at a bulb and tell whether or not it is defective. Find Probability distribution of the no. of defective bulbs, if four bulbs are drawn at random from this slot.

Sol^n:- Let X denote the number of defective bulbs in 4. Clearly X can take the values 0, 1, 2, 3 or 4.

no. of defective bulbs = 5

no. of good bulbs = 20

Total no. of bulbs = 25

$$P(X=0) = P(\text{no defective}) = P(\text{all good ones})$$

$$= \frac{{}^{20}C_4}{{}^{25}C_4} = \frac{20\times 19\times 18\times 17}{25\times 24\times 23\times 22} = \frac{969}{2530}$$

$$P(X=1) = P(\text{one defective \& 3 good ones}) = \frac{{}^{5}C_1\times {}^{20}C_3}{{}^{25}C_4} = \frac{1140}{2530}$$

$$P(X=2) = P(\text{2 defective \& 2 good ones}) = \frac{{}^{5}C_2\times {}^{20}C_2}{{}^{25}C_4} = \frac{380}{2530}$$

$$P(X=3) = P(\text{3 defective \& 1 good one}) = \frac{{}^{5}C_3\times {}^{20}C_1}{{}^{25}C_4} = \frac{40}{2530}$$

$$P(X=4) = P(\text{all 4 defective}) = \frac{{}^{5}C_4}{{}^{25}C_4} = \frac{1}{2530}$$

∴ The Probability distribution of the random variable X is

X:	0	1	2	3	4
$P(x)$:	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

Example: 2 A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and the variance of the no. of successes.

Sol^n:- Let X denote the no. of success. Clearly X can take the values 0, 1, 2 or 3.

Probability of success $= \frac{2}{6} = \frac{1}{3}$

" " failure $= 1 - \frac{1}{3} = \frac{2}{3}$

$P(X=0) = P(\text{no success}) = P(\text{all 3 failures}) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

$P(X=1) = P(\text{one success \& 2 failures}) = {}^3C_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$

$P(X=2) = P(\text{two success \& one failure}) = {}^3C_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$

$P(X=3) = P(\text{all 3 successes}) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

$\therefore$ The probability distribution of X is

X :	0	1	2	3
$P(X)$:	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

To find mean & variance

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	8/27	0	0
1	12/27	12/27	12/27
2	6/27	12/27	24/27
3	1/27	3/27	9/27

Mean $\mu = \Sigma p_i x_i = 1$

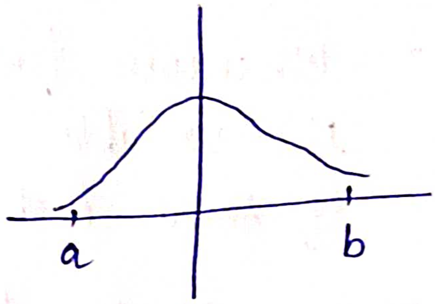
Variance $\sigma^2 = \Sigma p_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}$.

Continuous Random Variable :-

- A continuous random variable is one which can assume any value within an interval i.e all values of a continuous scale.

e.g → i.) The weights (in Kg) of a group of individuals
ii.) the heights of a group of individuals
iii.) Age of persons of a group.

$$\begin{array}{ccc} S_1 & \xrightarrow{X(S)} & 20 \\ S_2 & & 25 \\ S_3 & & 30 \\ \vdots & & \vdots \\ S_n & & 105 \end{array}$$



$[20, 105]$

Note:- The interval may be finite or infinite

Probability Density Function :- (PDF)

Let x be continuous random variable then PDF is defined as

$$P(a \le x \le b) = f(x) \text{ or } P(x) = \begin{cases} 0 & x < a \\ \phi(x) & a \le x \le b \\ 0 & x > b \end{cases}$$

Such that i.) $f(x) \geq 0$ ii.) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

→ For density function $f(x)$, the probability that the variate x falls in any interval (a, b) is given by

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

Note!! Any positive f^n of a variate x can be changed to give a probability density f^n $f(x)$ by multiplying it by a constant which will make the total area under the curve $y = f(x)$ equal to unity.

eg → we know $\int_0^2 x(2-x)\,dx = \frac{4}{3}$

hence if we multiply both sides by $\frac{3}{4}$, we get

$$\int_0^2 \frac{3}{4}x(2-x)\,dx = 1$$

Hence a Probability density f^n can be formed as given by

$$f(x) = \begin{cases} 0, & x<0 \\ \frac{3x(2-x)}{4}, & 0 \le x \le 2 \\ 0, & x>2 \end{cases}$$

Note:2 Whenever $f(x)$ is constant throughout its interval the variable is said to have a rectangular distribution of same Probability.

Example 1 - If the function $f(x)$ is defined by $f(x) = ce^{-x}$, $0 \le x \le \infty$, find the value of c which changes $f(x)$ to a Probability density f^n. (AKTU-2021)

Sol^n: - For $f(x)$ to be pdf we should have

a.) $f(x) \geq 0 \quad \forall x$ b.) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

So $f(x) = ce^{-x} \geq 0 \quad \forall x$

Since e^{-x} is always +ve $\forall x \in (0, \infty)$

Therefore $c \geq 0$

for second condition $\int_0^{\infty} ce^{-x}\,dx = 1$

i.e if $\left[-ce^{-x}\right]_0^{\infty} = 1$

$\Rightarrow -c\cancelto{0}{e^{-\infty}} + ce^{-0} = 1$

$\Rightarrow 0 + c = 1 \Rightarrow \boxed{c=1}$

Example:2 If $f(x)$ has Probability density cx^2, $0<x<1$, determine c and find the Probability that $\frac{1}{3} < x < \frac{1}{2}$ i.e $P\left(\frac{1}{3} < x < \frac{1}{2}\right)$

:- $f(x)$ will have a Probability density if $\int_0^1 cx^2\,dx = 1$

i.e $\left[\frac{1}{3}cx^3\right]_0^1 = 1$

$\Rightarrow \frac{1}{3}c - 0 = 1 \Rightarrow \frac{c}{3} = 1$

$\Rightarrow \boxed{c = 3}$

$P\left(\frac{1}{3} < x < \frac{1}{2}\right) = \int_{1/3}^{1/2} 3x^2\,dx = \left[x^3\right]_{1/3}^{1/2}$

$= \frac{1}{8} - \frac{1}{27} = \frac{19}{216}.$

Ex:-3. A random variable x has the density function

$f(x) = k.\frac{1}{1+x^2}, \quad -\infty < x < \infty$

Determine k and the Distribution function.

Solⁿ:- It will be a density fⁿ if

$\int_{-\infty}^{\infty} k.\frac{1}{1+x^2} = 1$

$\Rightarrow k.\left[\tan^{-1}x\right]_{-\infty}^{\infty} = 1$

$\Rightarrow k.\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$

$\Rightarrow k.\pi = 1 \Rightarrow \boxed{k = \frac{1}{\pi}}$

$F(x) = \int \frac{1}{\pi}.\frac{1}{1+x^2}\,dx = \frac{1}{\pi}\tan^{-1}x + C$

But $F(-\infty)$ should be zero for distribution fⁿ

$\therefore \frac{1}{\cancel{\pi}}\left(-\frac{\cancel{\pi}}{2}\right) + C = 0$

$\Rightarrow C = \frac{1}{2}$

$\therefore \boxed{F(x) = \frac{1}{\pi}\tan^{-1}x + \frac{1}{2},}$ for $-\infty < x < \infty.$

Mathematical Expectation for discrete Random Variable: r^{th}

Relation b/w expectation of a random variable x with moments →

if x is discrete random variable with probability distribution

$$x : x_1 \quad x_2 \quad \cdots \quad x_n$$
$$p(x) : p_1 \quad p_2 \quad \cdots \quad p_n \qquad \text{where } \sum_{i=1}^{n} p_i = 1$$

Then the expectation of x is denoted by $E(x)$ and defined as :-

$$E(x) = x_1 p_1 + x_2 p_2 + x_3 p_3 + \cdots + x_n p_n$$
$$E(x) = \sum_{i=1}^{n} x_i p_i \qquad \Sigma xp = \bar{x}$$

It is also known as "Mean value or Average value"
or
$$E(x) = \Sigma x \cdot p(x)$$

Remarks!- 1.
$$E(x^2) = \Sigma x^2 p(x)$$
$$E(x^3) = \Sigma x^3 p(x)$$
$$\vdots$$
$$E(x^r) = \Sigma x^r p(x) = \sum_{i=1}^{n} x_i^r p_i = \nu_r$$

r^{th} moment of discrete probability distribution about origin i.e $x=0$

$$\nu_r = \frac{\Sigma p_i x_i^r}{\Sigma p_i} = \Sigma p_i x_i^r = E(x^r) \qquad \{\because \Sigma p_i = 1$$

$$\mu_r' = \frac{\Sigma f(x-a)^r}{\Sigma f}$$

Some Important formulas:

1.) $E(C) = C$ where C is any constant

2.) $E(ax) = a\,E(x)$

3.) $E(x \pm y) = E(x) \pm E(y)$

4.) $E(xy) = E(x) \cdot E(y)$

r^{th} Moment about mean –

$$\mu_r = \frac{\sum p_i (x_i - \bar{x})^r}{\sum p_i}, \quad \sum p_i = 1$$

r^{th} moment about mean $\mu_r = \frac{\sum f (x - \bar{x})^r}{\sum f}$

$$\Rightarrow \mu_r = \sum p_i \cdot (x_i - \bar{x})^r$$

$$\boxed{\mu_r = E(x - \bar{x})^r}$$

$\bar{x} = E(x)$

Variance $r=1$ $\mu_1 = E(x - \bar{x}) = E(x) - \bar{x} E(1)$

$= \bar{x} - \bar{x} \cdot 1 = \bar{x} - \bar{x} = 0$

for $r=2$

Variance $\mu_2 = E(x - \bar{x})^2$

$= E(x^2 + \bar{x}^2 - 2\bar{x}x)$

$= E(x^2) + \bar{x}^2 E(1) - 2\bar{x} E(x)$

$= E(x^2) + \bar{x}^2 \cdot 1 - 2\bar{x} \cdot \bar{x}$

$= E(x^2) - \bar{x}^2 = E(x^2) - [E(x)]^2$

$$\boxed{\mu_2 = E(x^2) - [E(x)]^2} = \mu_2' - \mu_1'^2 = \sigma^2$$

standard deviation $= \sqrt{\text{Variance}} = \sqrt{\mu_2}$

Ex:1 A pair of coin is tossed, what is the expected value ?. Also find variance.

Solⁿ:- In tossing of two coins, The probability distributⁿ is

	TT	HT, TH	HH
x (no. of Heads) :	0	1	2
$p(x)$:	1/4	1/2	1/4

Now $E(x) = \sum x\, p(x)$

$= 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}$

$= 1$

Variance $\mu_2 = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum x^2 p(x)$$
$$= 0^2 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4}$$
$$= \frac{1}{2} + 1 = \frac{3}{2}$$

$\therefore$ variance $= \frac{3}{2} - 1^2 = \frac{1}{2}$

Ex:2 A pair of dice is thrown together, find the expected value.

Solⁿ:- In a throw of pair of dice, the probability distribution is

x (Sum of no. on dice)	Probability $p(x)$	$x\,p(x)$
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	20/36
6	5/36	30/36
7	6/36	42/36
8	5/36	40/36
9	4/36	36/36
10	3/36	30/36
11	2/36	22/36
12	1/36	12/36

$$\sum x\,p(x) = \frac{252}{36} = 7$$

Hence $E(x) = \sum x\,p(x) = 7$ Ans.

What is the expected value of the number of points that will be obtained in single throw with an ordinary die? Find variance also.

Sol^n:- Here the variate is the number of points showing on a die. It assumes the values 1, 2, 3, 4, 5, 6 with probability $\frac{1}{6}$ in each case.

Hence $E(x) = p_1x_1 + p_2x_2 + \dots + p_6x_6$

$$= \frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \frac{1}{6}\times 3 + \frac{1}{6}\times 4 + \frac{1}{6}\times 5 + \frac{1}{6}\times 6$$

$$= 3.5$$

Also $Var(x) = E(x^2) - [E(x)]^2 = \frac{1}{6}(1^2 + 2^2 + \dots + 6^2) - \left(\frac{7}{2}\right)^2$

$$= \frac{35}{12}.$$

Ex:- 4 In four tosses of a coin, let x be the no. of heads calculate the expected values of x.

Sol^n:- There will be all the heads, there is only one way

i.e $P(x=4) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{16}$

$$P(x=3) = \frac{{}^4C_3}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P(x=2) = \frac{{}^4C_2}{16} = \frac{6}{16} = \frac{3}{8}$$

$$P(x=1) = \frac{{}^4C_1}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P(x=0) = \frac{{}^4C_0}{16} = \frac{1}{16}$$

Hence the probability distribution of x is

x :	0	1	2	3	4
$p(x)$:	1/16	1/4	3/8	1/4	1/16

$\therefore E(x) = \sum_{x=0}^{4} x\,p(x) = 0\cdot\frac{1}{16} + 1\cdot\frac{1}{4} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{16}$

$$= 2.$$

Ex: 5. Find $E(x)$, $E(x^2)$, $E\{(x-\bar{x})^2\}$ for the following probability distribution:

x:	8	12	16	20	24
$p(x)$:	1/8	1/6	3/8	1/4	1/12.

Solⁿ:-

$$E(x) = \sum x\,p(x)$$

$$= 8\times\frac{1}{8} + 12\times\frac{1}{6} + 16\times\frac{3}{8} + 20\times\frac{1}{4} + 24\times\frac{1}{12}$$

$$= 1+2+6+5+2 = 16$$ (mean of distributⁿ)

Second moment about origin zero

$$E(x^2) = \sum x^2.p(x) = 8^2.\frac{1}{8} + 12^2.\frac{1}{6} + 16^2.\frac{3}{8} + 20^2.\frac{1}{4} + 24^2.\frac{1}{12}$$

$$= 276$$

variance of the distribution

$$E\{(x-\bar{x})^2\} = \sum (x-\bar{x})^2 p(x)$$

$$= (8-16)^2.\frac{1}{8} + (12-16)$$

Exper

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Expectation for continuous Random Variable :-

If x is a continuous random variable, then the expectation of x is

$$E(x) = \int_{-\infty}^{\infty} x\, p(x)\, dx \qquad \text{when } \boxed{\int_{-\infty}^{\infty} f(x)\, dx = 1}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 p(x)\, dx$$

$$\vdots$$

$$E(x^r) = \int_{-\infty}^{\infty} x^r p(x)\, dx$$

Note:-

1. $E(x) = \bar{x}$ = mean value / Avg value $= \mu$
2. $E(x^r) = \nu_r$ = rth moment about origin
3. $E(x-\bar{x})^r = \mu_r$ = moment about mean
4. $\mu_1 = 0$
5. $\mu_2 = E(x^2) - \{E(x)\}^2$ = Variance
6. S.D = $+\sqrt{\text{Variance}} = \sqrt{\mu_2}$

Ex:- A continuous random variable x has

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

represent the density, find the mean and variance and standard deviation of x.

Sol^n:- we have $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{-1} f(x)\, dx + \int_{-1}^{1} f(x)\, dx + \int_{1}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{-1} 0\, dx + \int_{-1}^{1} \frac{1}{2}(x+1)\, dx + \int_{1}^{\infty} 0\, dx = 1$$

$$\Rightarrow \frac{1}{2}\int_{-1}^{1} (x+1)\, dx = 1$$

$$\Rightarrow \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1} = 1$$

$$\frac{1}{2}\left[\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)\right] = 1$$

$$\frac{1}{2} \times 2 = 1 \Rightarrow 1 = 1$$

Hence f is P.D.F for x

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{-1} x \cdot 0\,dx + \int_{-1}^{1} x \cdot \frac{1}{2}(x+1)\,dx + \int_{1}^{\infty} x \cdot 0\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2 + x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} + \frac{1}{2}\right) - \left(-\frac{1}{3} + \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-1}^{1} x^2 f(x)\,dx$$

$$= \int_{-1}^{1} x^2 \cdot \frac{1}{2}(x+1)\,dx$$

$$= \frac{1}{3}$$

Hence Variance $\mu_2 = E(x^2) - [E(x)]^2$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

Standard deviation $= \sqrt{\mu_2} = \sqrt{\frac{2}{9}}$.

Binomial Probability Distribution (Binomial Distribution)

a Success is occurrence of the event

a failure is non-occurrence of the event.

Let there be n independent trials in an experiment. Let a random variable X denote the number of successes in these n trials. Let p be the probability of a success and q that of a failure in a single trial so that $p+q=1$. Let the trials be independent and p be constant for every trial.

Probability of r successes in n trials

r successes can be obtained in n trials in ${}^nC_r$ ways

$$P(r) = P(X=r) = {}^nC_r\, P(\underbrace{SSS\ldots S}_{r \text{ times}}\ \underbrace{FF\ldots F}_{(n-r) \text{ times}})$$

$$= {}^nC_r\, \underbrace{P(S)P(S)\ldots P(S)}_{r\text{-factors}}\ \underbrace{P(F)P(F)\ldots P(F)}_{n-r \text{ factors}}$$

$$= {}^nC_r\, \underbrace{p\,p\ldots p}_{r \text{ times}}\ \underbrace{q\cdot q\ldots q}_{n-r \text{ times}}$$

$$= {}^nC_r\, p^r q^{n-r}$$

Hence $$\boxed{P(r) = P(X=r) = {}^nC_r\, p^r q^{n-r}} \quad r=0,1,2,\ldots n$$
$$p+q=1$$

It is called Binomial Distribution.

Also $$\sum P(r) = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r}$$
$$= (q+p)^n$$
$$= 1^n$$
$$= 1$$

Binomial Thm : $(a+x)^n = \sum_{r=0}^{n} {}^nC_r\, a^r x^{n-r}$

Note:- If n independent trials constitute one experiment and this experiment is repeated N times & The frequency of r success is $N\,{}^nC_r p^r q^{n-r}$ & Binomial Distribution is $N(q+p)^n$.

Imp # Mean, Variance & Standard Deviation of Binomial Distribution :—

For the Binomial distribution, $P(r) = {}^nC_r q^{n-r} p^r,\ r = 0, 1, \ldots n$

Mean $\mu = E(r) = \sum_{r=0}^{n} r P(r)$ $\qquad E(x) = \sum x p(x)$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r q^{n-r} p^r$$

$$= \sum_{r=0}^{n} r \cdot \frac{n!}{r!(n-r)!} q^{n-r} p^r$$

$$= \sum_{r=0}^{n} \cancel{r} \cdot \frac{n(n-1)!}{\cancel{r}(r-1)!(n-r)!} q^{n-r} p^r$$

$$= np \sum_{r=1}^{n} \underbrace{\frac{(n-1)!}{(r-1)!(n-r)!}}_{{}^{n-1}C_{r-1}} q^{n-r} p^{r-1}$$

$$= np(q+p)^{n-1} \qquad \because p+q=1$$

$$= np$$

Hence $\boxed{\text{Mean } \mu = np}$

Variance $\sigma^2 = E(r^2) - \{E(r)\}^2$

$$= \sum_{r=0}^{n} r^2 p(r) - (np)^2$$

$$= \sum_{r=0}^{n} (r + r^2 - r) p(r) - (np)^2$$

$$= \sum_{r=0}^{n} r p(r) + \sum_{r=0} r(r-1) p(r) - (np)^2$$

$$= np + \sum_{r=0}^{n} r(r-1) p(r) - (np)^2 \quad \text{—} (1)$$

Now $\sum_{r=0}^{n} r(r-1)p(r) = \sum_{r=0}^{n} r(r-1)\, {}^nC_r q^{n-r} p^r$

$$= \sum_{r=0}^{n} r(r-1) \frac{n(n-1)(n-2)!}{r(r-1)(r-2)!\,(n-r)!} q^{n-r} p^r$$

$$= n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\,(n-r)!} p^{r-2} q^{n-r}$$

$$= n(n-1)p^2 (q+p)^{n-2}$$

$$= n(n-1)p^2 \qquad \because p+q=1$$

from ①

Variance $\sigma^2 = np + n(n-1)p^2 - n^2p^2$

$$= np - np^2$$

$$= np(1-p)$$

Hence $\boxed{\text{Variance } \sigma^2 = npq}$

$\boxed{S.D = \sqrt{npq}}$

Imp

⊕ Moment Generating function of Binomial Distribution :-

1. About origin →

$$M_r(t) = E(e^{tr}) = \sum_{r=0}^{n} e^{tr} p(r)$$

$$= \sum_{r=0}^{n} e^{tr}\, {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=0}^{n} {}^nC_r (pe^t)^r q^{n-r}$$

$$\boxed{M_r(t) = (q+pe^t)^n}$$

2. About Mean →

$$M_{r-np}(t) = E\{e^{t(r-np)}\}$$

$$= E\{e^{tr} \cdot e^{-npt}\} \qquad \because E(ax) = aE(x)$$

$$= e^{-npt} E(e^{tr})$$

$$\boxed{M_{r-np}(t) = e^{-npt} M_r(t)}$$

Recurrence or Recursion formula for the Binomial Distr.

For B.D, we have

$$P(r) = P(X=r) = {}^nC_r\, p^r q^{n-r} \quad \text{—(1)}$$

$$P(r+1) = P(X=r+1) = {}^nC_{r+1}\, p^{r+1} q^{n-(r+1)} \quad \text{—(2)}$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}\, p^{r+1}\, q^{n-(r+1)}}{{}^nC_r\, p^r\, q^{n-r}}$$

after solving

$$\boxed{P(r+1) = \frac{n-r}{r+1}\,\frac{p}{q}\,P(r)}$$

Moment about mean of binomial distribution

$$\mu_2 = npq, \quad \mu_3 = npq(q-p),$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq}$$

$$\gamma_1 = \frac{1-2p}{\sqrt{npq}}$$

So Measure of skewness of the binomial Distribution

$$\gamma_1 = \frac{1-2p}{\sqrt{npq}}$$

if $p < \frac{1}{2}$, skewness is +ve
if $p > \frac{1}{2}$, skewness is -ve
if $p = \frac{1}{2}$, it is zero

Measure of Kurtosis of the binomial Distribution

$$\beta_2 = 3 + \frac{1-6pq}{npq}$$

Ex:1 → Comment on the following statement:
for a Binomial distribution mean is 6 and variance is 9.

Solⁿ:–

Mean $\mu = np = 6$ —①

Variance $\sigma^2 = \mu_2 = npq = 9$ —②

dividing ① by ②

$$\frac{np}{npq} = \frac{6}{9}$$

$$\frac{1}{q} = \frac{2}{3} \Rightarrow q = \frac{3}{2} \Rightarrow q = 1.5$$

but $0 \le q \le 1$

∴ The given statement is false.

Ex:2 A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the success.

Solⁿ:–

Prob. of getting (1 or 6) on a toss

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore p = \frac{1}{3}$$

$$\& \ q = 1-p = 1-\frac{1}{3} = \frac{2}{3}$$

Here $n = 3$

i) mean $= np = 3 \times \frac{1}{3} = 1 = \mu$

ii) variance $\sigma^2 = npq = 3 \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{3}$

Ex:3. A binomial variable X satisfies the relation

$$9\,P(X=4) = P(X=2) \quad \text{when } n=6$$

Find the value of the parameter p and $P(X=1)$

Solⁿ:- Here $P(X=r) = {}^6C_r p^r q^{6-r}$ $(n=6)$ — ①

given $9P(X=4) = P(X=2)$ from ①

$\Rightarrow 9\, {}^6C_4 p^4 q^2 = {}^6C_2 p^2 q^4$

$\Rightarrow 9\, \cancel{{}^6C_2} p^4 q^2 = \cancel{{}^6C_2} p^2 q^4$ $\because {}^nC_r = {}^nC_{n-r}$

$\Rightarrow 9p^2 = q^2$ $\because p+q=1$

$\Rightarrow 9p^2 = (1-p)^2$

$\Rightarrow 9p^2 = 1+p^2-2p$

$\Rightarrow 8p^2+2p-1=0$

$\Rightarrow 8p^2+4p-2p-1=0$

$\Rightarrow 4p(2p+1)-1(2p+1)=0$

$\Rightarrow (4p-1)(2p+1)=0$

$\Rightarrow p=\frac{1}{4}$, $\boxed{p=-\frac{1}{2}}$ but $0 \le p \le 1$

so $\boxed{p=\frac{1}{4}}$ Ans $q = 1-\frac{1}{4} = \frac{3}{4}$

Hence $P(X=1) = {}^6C_1 p^1 q^5$

$= 6 \times \frac{1}{4} \times \left(\frac{3}{4}\right)^5$

$\boxed{P(X=1) = .3559}$ Ans

Ex:4 (Imp) Fit a binomial distribution to the following frequency data:

x:	0	1	2	3	4
f:	30	62	46	10	2

Solⁿ:-

f	x	fx
30	0	0
62	1	62
46	2	92
10	3	30
2	4	8
$\Sigma f = 150$		$\Sigma fx = 192$

Mean $\mu = \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{192}{150} = 1.28$

$\Rightarrow \boxed{\mu = 1.28}$

For Binomial distribution $\mu = np$

$np = 1.28$ (n is no. of trials)

$\Rightarrow 4p = 1.28$

$\Rightarrow \boxed{p = 0.32}$

& $q = 1 - p = 1 - 0.32 = 0.68$, $\boxed{q = 0.68}$

Also $N = \Sigma f = 150$

Hence Binomial distribution is

$$\boxed{N(q+p)^n = 150\,(0.68 + 0.32)^4}$$

Ex:-5 If 10% of the bolts produced by a machine are defective, determine the probability that out of 10 bolts chosen at random

i.) 1 ii.) None iii.) at most 2 bolts will be defective.

Sol^n:- Here $P(\text{defective}) = p = \frac{10}{100} = \frac{1}{10}$

$\therefore P(\text{non-defective}) = q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

Also, no. of bolts chosen, $n = 10$

The Probability of r defective bolts out of n bolts chosen at random is

$$P(r) = {}^nC_r\, p^r q^{n-r} = {}^{10}C_r\, p^r q^{10-r}$$

i.) $P(1) = {}^{10}C_1\, p^1 q^9 = 10 \cdot \frac{1}{10} \cdot \left(\frac{9}{10}\right)^9 = 0.3874$

ii.) Here $r = 0$

$P(0) = {}^{10}C_0\, p^0 q^{10} = 1 \cdot 1 \cdot \left(\frac{9}{10}\right)^{10} = 0.3486$

iii) Prob. that atmost 2 bolts will be defective

$$P(r \le 2) = P(0) + P(1) + P(2)$$

Now $P(2) = {}^{10}C_2 \, p^2 q^8$

$$= 45 \cdot \left(\frac{1}{10}\right)^2 \cdot \left(\frac{9}{10}\right)^8$$

$$= 0.1937$$

$\therefore$ $P(r \le 2) = 0.3486 + 0.3874 + 0.1937$

$$= 0.9297$$

Ex:-6 If the Probability of hitting a target is 10% and 10 shots are fired independently. What is the Probability that the target will be hit at least once? (2019)

Sol:- Here $p = \frac{10}{100} = \frac{1}{10}$, $q = 1-p = 1-\frac{1}{10} = \frac{9}{10}$

$$n = 10$$

Probability that the target will be hit at least once

$$P(r \ge 1) = 1 - P(r=0)$$

$$= 1 - {}^{10}C_0 \, p^0 q^{10-0}$$

$$= 1 - 1 \cdot 1 \cdot q^{10}$$

$$= 1 - \left(\frac{9}{10}\right)^{10}$$

Imp

$$= 0.6513$$

Ex:-7 Out of 800 families with 4 students children each, How many families would be expected to have i.) 2 boys and 2 girls

ii.) at least one boy

iii.) no girl

iv.) atmost two girls?

Assume equal Probabilities for boys and girls.

$\therefore$ Since Probabilities for boys and girls are ~~same~~ equal

So Prob. of having a boy $p = \frac{1}{2}$

Prob. of having a girl $q = 1 - \frac{1}{2} = \frac{1}{2}$

$n = 4$, $N = 800$

by binomial distribution $P(X=r) = {}^nC_r p^r q^{n-r}$

$N(q+p)^n = 800(q+p)^4$

i.) expected no. of families having 2 boys and 2 girls

$$N\,P(X=2) = 800\,{}^4C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^2$$

$$= 800 \times 6 \times \frac{1}{16} = 300.$$

ii.) The expected no. of families having atleast one boy

$$N\,P(X \geq 1) = 800[P(X=1) + P(X=2) + P(X=3) + P(X=4)]$$

$$= 800\left[{}^4C_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^3 + {}^4C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^2 + {}^4C_3 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^1 + {}^4C_4 \left(\tfrac{1}{2}\right)^4\right]$$

$$= 800 \times \frac{1}{16}\left[{}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4\right]$$

$$= 800 \times \frac{1}{16}[4 + 6 + 4 + 1]$$

$$= 750$$

iii.) The expected no. of families having no girl i.e 4 boys

$$N\,P(X=4) = 800\,{}^4C_4\, p^4 q^{4-4}$$

$$= 800 \times 1 \times \left(\tfrac{1}{2}\right)^4 \times \left(\tfrac{1}{2}\right)^0$$

$$= 50$$

iv.) The expected no. of families having at most two girls i.e having at least 2 boys

$$N\,P(X \geq 2) = 800[P(X=2) + P(X=3) + P(X=4)]$$

$$= 800 \times \frac{1}{16}[6 + 4 + 1] = \underline{550}$$

Ex:-8 A student is given a true false examination with 8 questions. If he corrects at least 7 questions, he passes the examination. Find the probability that he will pass, given that he guesses all questions.

Soln:- Here $n = 8$ (no. of questions asked)

$p = \frac{1}{2}$, $q = \frac{1}{2}$ (Since the question can either be true or false)

Probability that he will pass

$$P(r \geq 7) = P(r = 7) + P(r = 8)$$
$$= {}^8C_7 \, p^7 q^1 + {}^8C_8 \, p^8 q^0$$
$$= 8 \cdot \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right) + 1 \cdot \left(\frac{1}{2}\right)^8 \cdot 1$$
$$= \frac{1}{2^8} [8 + 1]$$
$$= 0.3516$$

Ex:- 9 The prob. of a man hitting a target is $\frac{1}{3}$. How many times must he fire so that the probability of his hitting the target at least once is more than 90%

Soln:- Here $p = \frac{1}{3}$

The prob. of not hitting the target in n trials is q^n.

Therefore, to find the smallest n for which the probability of hitting at least once is more than 90%, we have

$$1 - q^n > 0.9$$
$$\Rightarrow 1 - \left(\frac{2}{3}\right)^n > 0.9 \Rightarrow \left(\frac{2}{3}\right)^n < 0.1$$

The smallest n for which the above inequality holds true is 6 Hence he must fire 6 times.

Poisson Distribution

Poisson Distribution as a limiting case of Binomial Distribution :→

→ If the parameters n and p of a binomial distribution are known, we can find distribution.

→ But in situations where n is very large and p is very small, application of binomial distribution is very laborious.

→ However, if we assume that as $n \to \infty$ and $p \to 0$ such that np always remains finite (say λ), we get the Poisson approximation to the Binomial distribution. Thus $\lambda = np$

Now, for a binomial distribution

$$P(X=r) = {}^nC_r \, p^r q^{n-r}$$

$$= \frac{n!}{r!\,(n-r)!} p^r (1-p)^{n-r} \qquad (\because p+q=1)$$

$$= \frac{n(n-1)(n-2)\cdots(n-(r-1))\cancel{(n-r)!}}{r!\,\cancel{(n-r)!}} \left(\frac{\lambda}{n}\right)^r \left(1-\frac{\lambda}{n}\right)^{n-r}$$

$(\because np = \lambda$
$\therefore p = \frac{\lambda}{n})$

$$= \frac{\lambda^r}{r!} \frac{n(n-1)(n-2)\cdots(n-r+1)}{n^r} \left(1-\frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{\lambda^r}{r!} \frac{n(n-1)(n-2)\cdots(n-r+1)}{n^r} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!} \left(\frac{n}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}$$

—(1)

taking $n \to \infty$ in eqn (1), each of $(r-1)$ factors $\left(1-\frac{1}{n}\right), \left(1-\frac{2}{n}\right), \ldots, \left(1-\frac{r-1}{n}\right) \to 1$ and $\left(1-\frac{\lambda}{n}\right)^{r} \to 1$

Since $\lim\limits_{x \to \pm\infty} \left(1+\frac{1}{x}\right)^{x} = e$, So $\left[\left(1-\frac{\lambda}{n}\right)^{\frac{-n}{\lambda}}\right]^{-\lambda} \to e^{-\lambda}$ as $n \to \infty$

Hence $$\boxed{P(X=r) = \frac{e^{-\lambda}\lambda^{r}}{r!}} \quad (r = 0, 1, 2, \ldots)$$

where λ is finite no. $= np$
which is called a probability distribution
& hence Poisson Probability distribution.

Note: (1) λ is called the parameter of distribution
(2) The sum of the Probabilities $P(r)$ for $r = 0, 1, 2, \ldots$ is 1
i.e $\sum P(r) = 1$, $r = 0, 1, 2, \ldots$

Since $P(0) + P(1) + P(2) + P(3) + \cdots$

$$= e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \cdots$$

$$= e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right)$$

$$= e^{-\lambda} \cdot e^{\lambda}$$

$$= 1$$

Recurrence formula for the Poisson Distribution

for P.D, $P(r) = \frac{e^{-\lambda}\lambda^{r}}{r!}$ and $P(r+1) = \frac{e^{-\lambda}\lambda^{r+1}}{(r+1)!}$

$$\therefore \frac{P(r+1)}{P(r)} = \frac{\lambda r!}{(r+1)!} = \frac{\lambda}{r+1}$$

So $$\boxed{P(r+1) = \frac{\lambda}{r+1} P(r)}, \quad r = 0, 1, 2, \ldots$$

This is called Recurrence or recursion formula for P.D.

Mean and Variance of the Poisson Distribution :→

For P. D. $P(r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$

Mean $\mu = \sum_{r=0}^{\infty} r\,P(r) = \sum_{r=0}^{\infty} r\cdot\dfrac{e^{-\lambda}\lambda^r}{r!}$

$$= e^{-\lambda}\sum_{r=1}^{\infty}\frac{\lambda^r}{(r-1)!}$$

$$= e^{-\lambda}\left(\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \cdots\right)$$

$$= e^{-\lambda}\cdot\lambda\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right)$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

Hence Mean of the Poisson Distribution $\boxed{\mu = \lambda = np}$

Variance $\sigma^2 = E(r^2) - \{E(r)\}^2$ $\quad (\lambda = E(r))$

$$= \sum_{r=0}^{n} r^2 P(r) - \lambda^2$$

$$= \sum_{r=0}^{n} (r + r^2 - r)\,P(r) - \lambda^2 \qquad \because E(x+y) = E(x) + E(y)$$

$$= \sum_{r=0}^{n} r\,P(r) + \sum_{r=0}^{n} r(r-1)P(r) - \lambda^2$$

$$= \lambda + \sum_{r=0}^{n} r(r-1)P(r) - \lambda^2 \quad \text{—— (2)}$$

Now $\sum_{r=0}^{n} r(r-1)P(r) = \sum_{r=0}^{n} r(r-1)\dfrac{e^{-\lambda}\lambda^r}{r!}$

$$= \sum_{r=0}^{n} \cancel{r(r-1)}\frac{e^{-\lambda}\lambda^r}{\cancel{r(r-1)}(r-2)!}$$

$$= \sum_{r=2}^{n}\frac{e^{-\lambda}\lambda^r}{(r-2)!} = \lambda^2 e^{-\lambda}\sum_{r=2}^{n}\frac{\lambda^{r-2}}{(r-2)!}$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

from eq^n (2)

variance $\sigma^2 = \lambda + \cancel{\lambda^2} - \cancel{\lambda^2}$

$= \lambda$

Hence variance of P.D

$$\boxed{\sigma^2 = \lambda = np}$$

$$\boxed{\text{Mean } \mu = \lambda = np = \text{variance } \sigma^2}$$

Mode of Poisson Distribution :-

Let $P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}$, $r = 0, 1, 2, \ldots$

The value of r which has a greater Probability than any other value of r is the mode of the Poisson distribution.

Let r is the mode of P.D, Then

$P(X=r) \geq P(X=r+1)$ & $P(X=r) \geq P(X=r-1)$

$\Rightarrow \frac{e^{-\lambda}\lambda^r}{r!} \geq \frac{e^{-\lambda}\lambda^{r+1}}{(r+1)!}$ and $\frac{e^{-\lambda}\lambda^r}{r!} \geq \frac{e^{-\lambda}\lambda^{r-1}}{(r-1)!}$

$\Rightarrow 1 \geq \frac{\lambda}{r+1}$ and $\frac{\lambda}{r} \geq 1$

$\Rightarrow r+1 \geq \lambda$ and $r \leq \lambda$

$\Rightarrow r \geq \lambda - 1$ and $r \leq \lambda$

i.e $\boxed{\lambda - 1 \leq r \leq \lambda}$

Case 1:- If λ is +ve Integer, there are two modes $\lambda - 1$ and λ

Case 2:- If λ is in fractional form, there is one mode and is the integer value b/w $1-\lambda$ and λ.

Ex-1 In a Poisson distribution $P(r)$ for $r=0$ is 10%. Find the mean.

Sol^n:-
$$P(r) = \frac{e^{-\lambda}\lambda^r}{r!} \quad \text{—①}$$

given $P(0) = 10\% = \frac{10}{100} = \frac{1}{10}$

Then $\frac{e^{-\lambda}\lambda^0}{0!} = \frac{1}{10}$

$\Rightarrow e^{-\lambda} = \frac{1}{10} \Rightarrow e^{\lambda} = 10 \Rightarrow \lambda = \log_e 10 = 1$

So mean $\mu = \lambda = 1$

Ex:-2 Using Poisson distribution, find the Probability that the ace of the spades will be drawn from a pack of well-shuffled cards at least once in 104 consecutive trials.

Sol^n:-
Here $p = \frac{1}{52}$, $n = 104$

$\lambda = np = 104 \times \frac{1}{52} = 2$

Prob (at least once) $= P(r \geq 1)$

$= 1 - P(0)$

$= 1 - \frac{e^{-\lambda}.\lambda^0}{0!}$

$= 1 - e^{-2}$

$= 1 - 0.135335$

$\simeq 0.8647$

Ex:3 If X is a Poisson variate and $P(X=1) = P(X=2)$. Find $P(X=4)$.

Sol^n:- For P.D $P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}$ $r = 0, 1, 2, \ldots$

given $P(X=1) = P(X=2)$

$\Rightarrow \frac{\cancel{e^{-\lambda}}\lambda}{1!} = \frac{\cancel{e^{-\lambda}}\lambda^2}{2!} \Rightarrow \lambda = \frac{\lambda^2}{2} \Rightarrow \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda = 0, 2$

So $P(X=4) = \frac{e^{-\lambda}\lambda^4}{4!} = 0$ (for $\lambda = 0$)

$P(X=4) = \frac{e^{-2}2^4}{4!} = \frac{e^{-2}.16}{24}$ (for $\lambda = 2$)

$= \frac{2e^{-2}}{3}$

Ex:- 4 Imp Fit a Poisson distribution to the following data and calculate theoretical frequencies:

Deaths :	0	1	2	3	4
Frequencies :	122	60	15	2	1

Soln:-

x	f	fx
0	122	0
1	60	60
2	15	30
3	2	6
4	1	4
	$\Sigma f = 200$	$\Sigma fx = 100$

Mean of given distribution $= \frac{\Sigma fx}{\Sigma f} = \frac{100}{200} = \frac{1}{2} = 0.5$

Required Poisson distribution $= N.\frac{e^{-\lambda}\lambda^r}{r!} = 200 . \frac{e^{-.5}(.5)^r}{r!}$

$= (121.306)\frac{(0.5)^r}{r!}$

r	$N.P(r)$	Theoretical frequency
0	$121.306 \times \frac{(0.5)^0}{0!} = 121.306$	121
1	$121.306 \times \frac{(0.5)^1}{1!} = 60.653$	61
2	$121.306 \times \frac{(0.5)^2}{2!} = 15.163$	15
3	$121.306 \times \frac{(0.5)^3}{3!} = 2.527$	3
4	$121.306 \times \frac{(0.5)^4}{4!} = 0.3159$	0
		Total = 200

5 i). If the Probabilities of a bad reaction from a certain injection is 0.0002, determine the chance that out of 1000 individuals more than two will get a bad reacth.

Sol^n:- Here $p = 0.00002$, $n = 1000$

$\therefore \lambda = np = 1000 \times 0.0002 = 0.2$

Since the Prob. of bad reacth is very small and no. of trials is very high, we use Poisson distribution here.

The Prob. that out of 1000 individuals, more than 2 will get a bad reaction $= P(r>2)$

$$= 1 - P(r \leq 2)$$

$$= 1 - [P(0) + P(1) + P(2)] \quad \text{---} (1)$$

Now $P(0) = \dfrac{e^{-0.2}(0.2)^0}{0!} = 0.8187$

$P(1) = \dfrac{e^{-0.2}(0.2)^1}{1!} = 0.1637$

$P(2) = \dfrac{e^{-0.2}(0.2)^2}{2!} = 0.0164$

from (1), we have

Required Prob. $= 1 - [0.8187 + 0.1637 + 0.0164]$

$= 0.0012$.

ii) The Prob. that a man aged 50 years will die within a year is 0.01125. What is the Probability that of 12 such men, at least 11 will reach their 51st birthday?

Sol^n:- Here, $p = 0.01125$, $n = 12$

So $\lambda = np = 12 \times 0.01125 = 0.135$

P(at least 11 survive) = P(at most 1 die) $= P(\leq 1)$

$$= P(0) + P(1)$$

$$= \frac{e^{-\lambda}\cdot\lambda^0}{0!} + \frac{e^{-\lambda}\lambda}{1!}$$

$$= e^{-.135}(1 + 0.135)$$

$$= 1.135 \times 0.87366 = 0.9916$$

Ex:-6 Six coins are tossed 6400 times. Using P.D, determine the approximate probability of getting six heads x times.

Solⁿ:- Prob. of getting one head with one coin $= \frac{1}{2}$

$\therefore$ The Prob. of getting six heads with six coins $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$$p = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$n = 6400$$

$\therefore$ Average no. of six heads with six coins in 6400 throws

$$= np = 6400 \times \frac{1}{64} = 100$$

$\therefore$ mean of Poisson distribution $\lambda = 100$

Approximate Probability of getting six heads x times in P.D $= \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-100}(100)^x}{x!}$

Normal Distribution :-

The Normal distribution is a continuous distribution. It can be derived from the binomial distribution in the limiting case when n, no. of trials is very large and p, the Probability of a success, is close to $\frac{1}{2}$.

A continuous random variable X is said to have a normal distribution with parameters μ (called mean) and σ^2 (called variance) if Probability density fn is given by

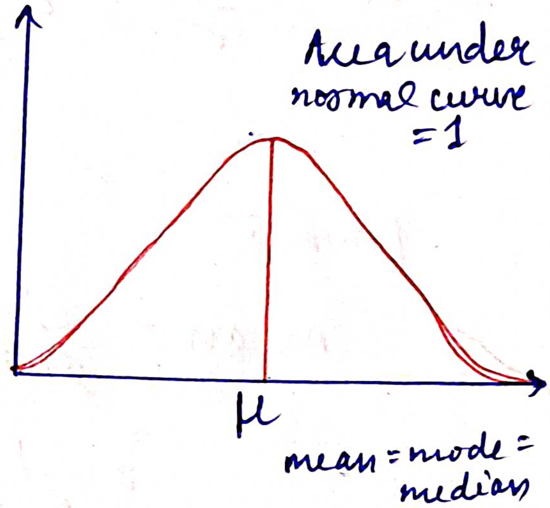
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

such that

i) $f(x) \geq 0 \;\; \forall \; x$

ii.) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e ii) the total area under the normal curve above x-axis is 1



iii.) normal distribution is Symmetric about mean.

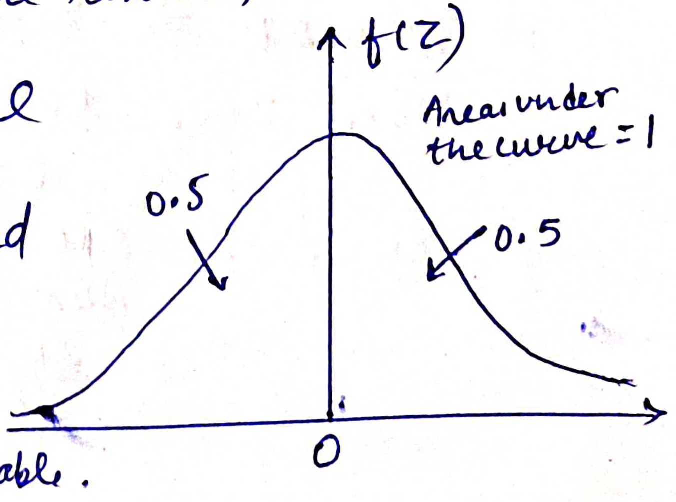
iii) The mean, mode & median of the distribution coincide.

Standard form of the Normal distribution :-

If X is a normal random variable with mean μ and standard deviation σ, then the random variable

$$\boxed{Z = \frac{X-\mu}{\sigma}}$$

has the normal distribution with mean 0 and S.D 1. The random variable Z is called the standardized (or standard) normal random variable.



The Probability density f^n for the normal distribution in standard form is given by

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}}$$

Mean of the Normal Distribution :→

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{Mean} = \mu' = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad — (1)$$

Let $\frac{x-\mu}{\sigma} = z \Rightarrow x = \mu + \sigma z$

$\Rightarrow dx = \sigma dz$

when $x = -\infty$, $z = -\infty$

$x = \infty$, $z = \infty$

from (1)

$$\text{mean} = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}z^2} dz + \sigma \int_{-\infty}^{\infty} \underbrace{z e^{-\frac{1}{2}z^2}}_{\text{odd } f^n} dz \right]$$

$$\underbrace{\qquad}_{=0}$$

$$= \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2} dz$$

$\left[\because \int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx \text{ if } f(x) \text{ is even } f^n\right]$

Let $\frac{z^2}{2} = t$

$\Rightarrow z^2 = 2t \Rightarrow z = \sqrt{2}\sqrt{t} \Rightarrow dz = \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{t}} dt = \frac{1}{\sqrt{2}\sqrt{t}} dt$

when $z = 0$, $t = 0$

$z = \infty$, $t = \infty$

$$\text{So Mean} = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-t} \cdot \frac{1}{\sqrt{2}\sqrt{t}} dt$$

$$= \frac{\cancel{2}\mu}{\cancel{2}\sqrt{\pi}} \int_0^{\infty} e^{-t} \cdot \frac{1}{\sqrt{t}} dt = \frac{\mu}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$\text{Mean} = \frac{\mu}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt \qquad \left[\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt \right]$$

$$= \frac{\mu}{\sqrt{\pi}} \cdot \Gamma\tfrac{1}{2}$$

$$\because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$= \frac{\mu}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}}$$

$$= \mu$$

Hence $\boxed{\text{Mean } \bar{x} = \mu}$

Moments about mean :- $\boxed{\mu_r = E(x-\bar{x})^r}$

i.) first moment about mean $\mu_1 = 0$ (always)

ii.) $\mu_2 = E(x-\bar{x})^2 =$ 2nd moment about mean

$$\text{variance} = E(x-\bar{x})^2 \qquad \bar{x} = \mu$$

$$= E(x-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Putting
$\frac{x-\mu}{\sigma} = z$
$x-\mu = \sigma z$
$x = \sigma z + \mu$
$dx = \sigma dz$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^2 e^{-\frac{1}{2}z^2} \cancel{\sigma}\, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z^2 e^{-\frac{1}{2}z^2}}_{\text{even } f^n} dz$$

let $\frac{1}{2}z^2 = t$
$dz = \frac{1}{\sqrt{2}} t^{\frac{1}{2}-1} dt$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty z^2 e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\cancel{\sqrt{2}}\sigma^2}{\sqrt{\pi}} \int_0^\infty (2t)\, e^{-t} \frac{1}{\cancel{\sqrt{2}}} t^{\frac{1}{2}-1} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{\frac{1}{2}} dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{\frac{3}{2}-1} dt$$

$$\text{variance} = \frac{2\sigma^2}{\sqrt{\pi}} \sqrt{\tfrac{3}{2}} \qquad \sqrt{n} = \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \cdot \frac{1}{\cancel{2}} \cancel{\sqrt{\pi}}$$

$$= \sigma^2$$

$$\boxed{\text{Variance} = \mu_2 = \sigma^2 = 2^{nd} \text{ moment about mean}}$$

and $\boxed{\mu_3 = 0}$

Ex:-1 The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. if 5000 pairs are insured, how many pairs would be expected to need replacement after 12 months. (Given $P(z \geq 2) = 0.0228$) (AKTU-2018)

Soln:- Given Mean $\mu = 8$, standard deviation $\sigma = 2$

Number of pairs of shoes $N = 5000$

Total months $x = 12$

When $x = 12$, $z = \frac{x-\mu}{\sigma} = \frac{12-8}{2} = 2$

$P(x \geq 12) = P(z \geq 2) = 0.0228$

Number of pairs whose life is more than 12 months

$N \times P(x \geq 12) = 5000 \times 0.0228$

$= 114$

pair of shoes needing replacement after 12 months

$= 500 - 114$

$= 4886$

Q.2. Assume mean height of soldiers to be 68.22 inches with a variance of 10.8 inches square. How many soldiers in a regiment of 1,000 would you expect to be over 6 feet tall, given that the area under the standard normal curve b/w $z=0$ and $z=0.35$ is 0.1368 & b/w $z=0$ and $z=1.15$ is 0.3746.

Soln:- Here $\mu = 68.22$, $\sigma^2 = 10.8 \Rightarrow \sigma = \sqrt{10.8}$

Let $x = 6$ feets $= 6 \times 12$ inches
$= 72$ inches

$$z = \frac{x-\mu}{\sigma} = \frac{72-68.22}{\sqrt{10.8}} = 1.15$$

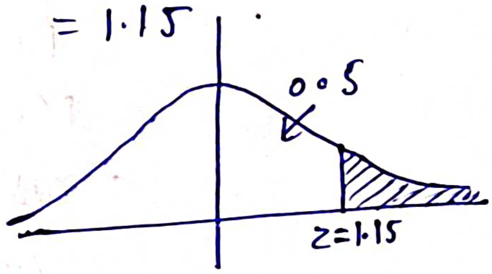
$P(x > 72)$

$= P(z > 1.15)$

$= P(0 \le z < \infty) - P(0 \le z \le 1.15)$

$= 0.5 - 0.3746$

$= 0.1254$



$\therefore$ Expected no. of soldiers
$= 1000 \times 0.1254$
$= 125.4 \simeq 125$ (app.)

Ex:3 A sample of 100 dry battery cells tested to find the length of life produced the following results:
$\bar{x} = 12$ hours, $\sigma = 3$ hours
Assuming the data to be normally distributed, what percentage of battery cells are expected to have life i) more than 15 hours
ii.) less than 6 hours
iii.) b/w 10 and 14 hours (AKTU-2018)

Soln:- Let x denotes the length of life of dry battery cells.

given $\bar{x} = \mu = 12$, $\sigma = 3$

we have $z = \frac{x-\mu}{\sigma} = \frac{x-12}{3}$

i.) when $x = 15$, $z = \frac{15-12}{3} = \frac{3}{3} = 1$

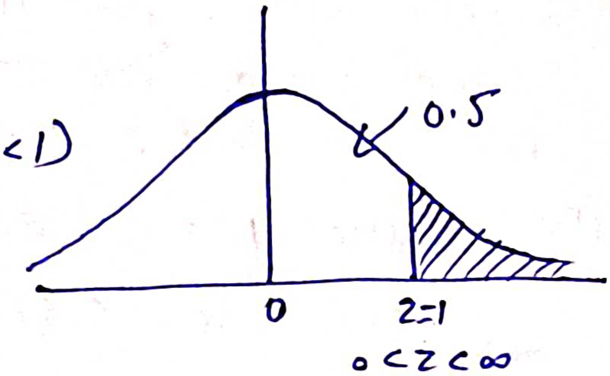
$\therefore P(x > 15) = P(z > 1)$

$= P(0 < z < \infty) - P(0 < z < 1)$

$= 0.5 - 0.3413$

$= 0.1587$

$= 15.87\%$



ii.) when $x = 6$, $z = \frac{6-12}{3} = \frac{-6}{3} = -2$

$\therefore P(x < 6) = P(z < -2)$

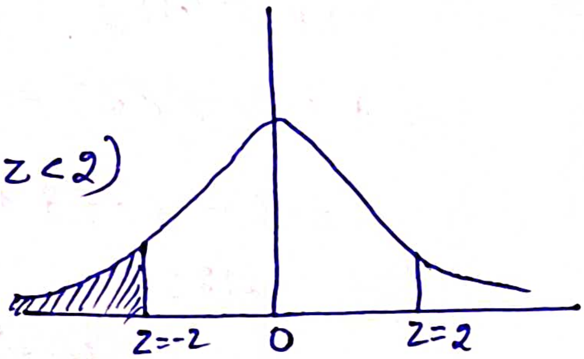
$= P(z > 2)$

$= P(0 < z < \infty) - P(0 < z < 2)$

$= 0.5 - 0.4772$

$= 0.0228$

$= 2.28\%$



iii.) when $x = 10$, $z = \frac{10-12}{3} = \frac{-2}{3} = -0.67$

when $x = 14$, $z = \frac{14-12}{3} = \frac{2}{3} = 0.67$

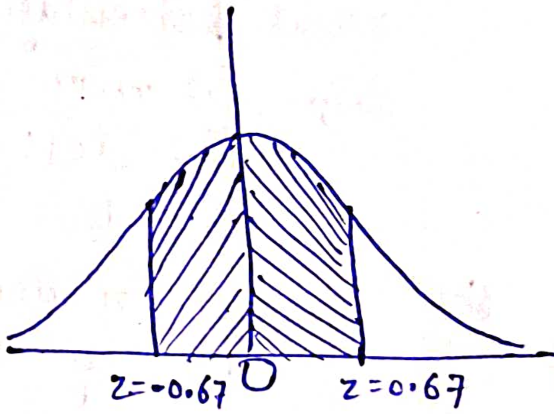
$P(10 < x < 14) = P(-0.67 < z < 0.67)$

$= 2P(0 < z < 0.67)$

$= 2 \times 0.2485$

$= 0.4970$

$= 49.70\%$



Ex:-4 In a Sample of 1000 cases, the mean of a certain test is 14 and S.D is 2.5. Assuming the distributn to be normal, find

i.) how many students score b/w 12 and 15?

ii.) how many score above 18?

iii.) how many score below 8?

iv.) how many score 16?

Soln:- Given $N = 1000$, $\mu = 14$, $\sigma = 2.5$

i.) let x = no. of students , $z = \frac{x-\mu}{\sigma}$

when $x = 12$, $z = \frac{12-14}{2.5} = -0.8$

$x = 15$, $z = \frac{15-14}{2.5} = 0.4$

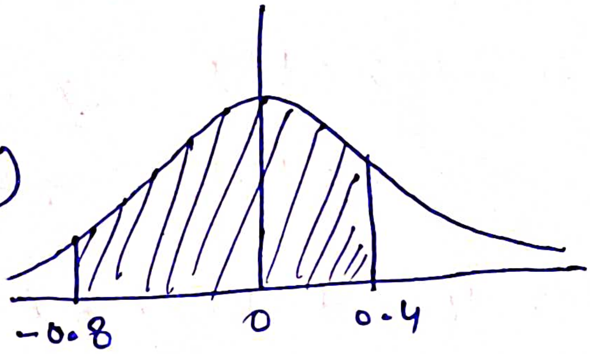
$P(12 < x < 15)$

$= P(-0.8 < z < 0.4)$

$= P(0 < z < 0.8) + P(0 < z < 0.4)$

$= 0.2881 + 0.1554$

$= 0.4435$



Required no. of students $= 1000 \times 0.4435$

$= 444$ (app.)

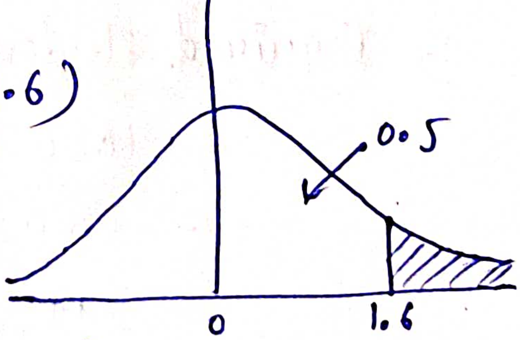
ii.) when $x = 18$, $z = \frac{18-14}{2.5} = 1.6$

$P(x > 18) = P(z > 1.6)$

$= P(0 < z < \infty) - P(0 < z < 1.6)$

$= 0.5 - 0.4452$

$= 0.0548$



Required no. of students $= 1000\, P(x > 18)$

$= 1000 \times 0.0548 = 54.8 = 55$ (app.)

iii.) When $x = 8$, $z = \frac{8-14}{2.5} = -2.4$

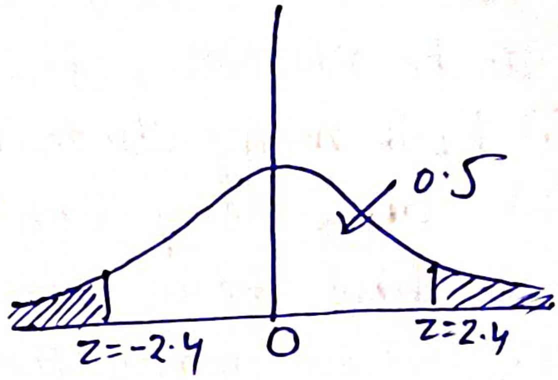
$P(x<8) = P(z<-2.4)$

$= P(z>2.4)$

$= 0.5 - P(0<z<2.4)$

$= 0.5 - 0.4918$

$= 0.0082$



$\therefore$ Required no. of students $= N\,P(x<8)$

$= 1000 \times 0.0082$

$= 8.2 \simeq 8$ (app.)

(iv.) When $x = 60$, $z = \frac{60-14}{2.5} = \frac{46}{2.5}$

iv.) $\because$ $15.5 < 16 < 16.5$

So let $x = 15.5$, $z = \frac{15.5-14}{2.5} = 0.6$

& $x = 16.5$, $z = \frac{16.5-14}{2.5} = 1$

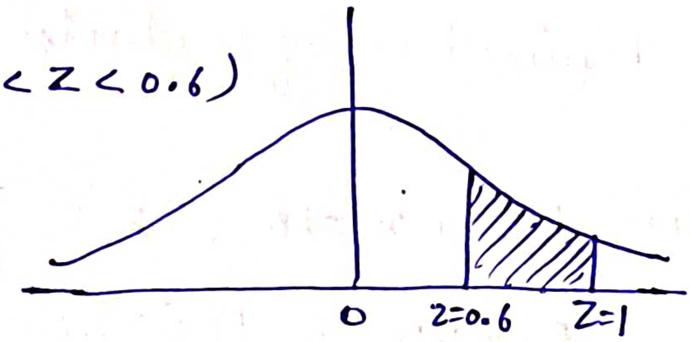
$P(x=16) = P(15.5 < x < 16.5)$

$= P(0.6 < z < 1)$

$= P(0<z<1) - P(0<z<0.6)$

$= 0.3413 - 0.2257$

$= 0.1156$



$15.5 < 16 < 16.5$

$\therefore$ Required students

$= 1000\,P(x=16)$

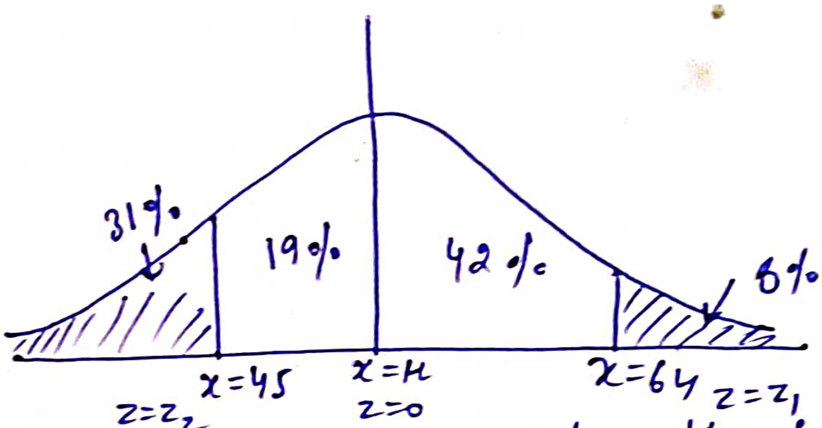
$= 1000 \times 0.1156$

$= 115.6 \simeq 116$ (app.)

Ex:-5. In a Normal distribution, 31% of the items are under 45 and 8% are over 64. find the mean and S.D of the distribution
It is given that if $f(t) = \frac{1}{\sqrt{2\pi}} \int_0^t e^{-\frac{1}{2}x^2} dx$ then

$f(0.5) = 0.19$ and $f(1.4) = 0.42$

Sol^n:-



Let μ be the mean & σ be the S.D of the distribution

Then $z = \frac{x-\mu}{\sigma}$ — (1)

i.) At $x=64$, let $z=z_1$

Then from (1), $z_1 = \frac{64-\mu}{\sigma}$

Area from $z=0$ to $z=z_1$
$= 42\%$
$= \frac{42}{100} = 0.42$

$\Rightarrow 1.4 = \frac{64-\mu}{\sigma}$

$\Rightarrow 64-\mu = 1.4\sigma$. — (2)

ii.) at $x=45$, let $z=z_2$
from (1)

$z_2 = \frac{45-\mu}{\sigma}$

Area from $z=z_2$ to $z=0$
$= 19\%$
$= \frac{19}{100}$
$= .19$

$\Rightarrow -0.5 = \frac{45-\mu}{\sigma}$

$\Rightarrow 45-\mu = -0.5\sigma$ — (3)

on solving (2) & (3), $\boxed{\sigma = 10}$ ← S.D

$\boxed{\mu = 50}$ ← Mean